JOURNAL OF COMPUTATIONAL PHYSICS 69, 482–486 (1987)

Note

An Alternative Form for the Legendre Polynomial Expansion Coefficients

1. Introduction

For $-1 \leqslant x \leqslant 1$, the expansion of a function $f(x)$ as a sum of Legendre polynomials $P_j(x)$ with coefficient a_j is given by

$$f(x) = \tfrac{1}{2} \sum_{j=0}^{j=\infty} (2j+1)\, a_j P_j(x), \tag{1}$$

$$a_j = \int_{-1}^{+1} f(x)\, P_j(x)\, dx. \tag{2}$$

The analytic evaluation of the coefficients a_j requires the specific expression for the kth Legendre polynomial for all values of $k \leqslant j$ and, in general, requires a sum of integrals of the form $\int x^k f(x)\, dx$.

A new form for the coefficients $A_j\ (=a_j)$ is given by,

$$A_j = \frac{2^{1/2}}{\pi} \int_{-1}^{+1} \frac{dy \cos[(j+1/2)\cos^{-1} y]}{(1-y^2)^{1/2}}\, L(y), \tag{3}$$

where

$$L(y) = \int_{-1}^{y} \frac{f(x)\, dx}{(y-x)^{1/2}}. \tag{4}$$

This form has a number of interesting properties including the one that the same integral involving $f(x)$ is used for all coefficients. In addition, the Legendre polynomials themselves do not appear explicity in A_j.

The proof of the equivalence of A_j and a_j is given in Section 2. Certain features of A_j are illustrated in Section 3 by a calculation of the expansion coefficients for a probability function derived from the Born approximation to the problem of MeV-electrons scattered by a screened Coulomb potential.

0021-9991/87 $3.00

2. Equivalence of a and A

For an arbitrary coefficient A_n, substitution of Eq. (1) for $f(x)$ into Eq. (3) gives (with $\phi = \cos^{-1} y$ and some rearrangement),

$$A_n = \frac{2^{1/2}}{\pi} \sum_{j=0}^{\infty} \{j + \tfrac{1}{2}\} a_j \cdot \int_{-1}^{1} \frac{\cos(n + \frac{1}{2})\phi}{(1-y^2)^{1/2}} dy \cdot \int_{-1}^{y} \frac{P_j(x)\, dx}{(y-x)^{1/2}}. \tag{5}$$

From Gradshteyn and Ryzhik [2, Sect. 7–225–1],

$$\int_{-1}^{y} \frac{P_j(x)\, dx}{(y-x)^{1/2}} = \frac{1}{(j+\frac{1}{2})} \frac{1}{(1+y)^{1/2}} \cdot [T_j(y) + T_{j+1}(y)],$$

where $T_j(y)$ is a Chebyshev polynomial of the first kind. If the cosine function in Eq. (5) is expressed in terms of the Chebyshev polynomials, then the orthogonality conditions on these polynomials result in a value of π for $j = n$ in the sum of integrals and zero for all other j values.

3. Evaluation of A_j

The procedure for evaluating the expansion coefficients A_j will be illustrated by considering the expansion of a function that appears in the theory of electron scattering. For an electron scattered by a screened Coloumb potential, the scattering probability per unit solid angle at θ is given in the Born approximation by $f(x)/2\pi$, where

$$f(x) = \frac{B(B+2)}{2} \frac{1}{(1+B-x)^2} \tag{6}$$

and

$$\int_{-1}^{+1} f(x)\, dx = 1.$$

In these expressions $x = \cos\theta$ and the constant B, which depends on the total energy, is related to the total scattering cross section σ_T by,

$$\sigma_T = \frac{2}{B(B+2)}.$$

For all A_j coefficients, the function $L(y)$ must be determined. If $f(x)$ is given by Eq. (6), we have

$$L(y) = \frac{B(1+y)^{1/2}}{(1+B-y)} + \frac{B(B+2)}{(1+B-y)^{3/2}} \arctan\left[\frac{1+y}{1+B-y}\right]^{1/2}.$$

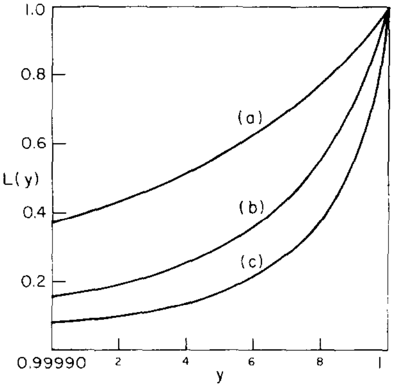

FIG. 1. The function $L(y)$ given by Eq. (4) for different values of the energy parameter B in $f(x)$ (Eq. (6)). (a) $B = 10.8 \times 10^{-5}$, (b) $B = 4.05 \times 10^{-5}$, (c) $B = 2.16 \times 10^{-5}$. All the $L(y)$ functions are zero at $y = -1$, and have been normalized to unity at $y = +1$. For $a-c$, the values of B correspond to the scattering by aluminum of electrons with energies 0.5, 1.0, and 1.5 MeV, respectively.

Figure 1 shows several curves for $L(y)$ with different values for the constant B. For illustration purposes, the $L(y)$ values have been normalized to unity at $x = 1$ and, since the curves are very steep, only the portion near $x = 1$ is shown. All $L(y)$ curves decrease monotonically to zero at $x = -1$.

If we now write

$$F(y) = \cos[(j + \tfrac{1}{2}) \cos^{-1} y]\, L(y),$$

then

$$A_j = \frac{2^{1/2}}{\pi} \int_{-1}^{+1} \frac{F(y)\, dy}{(1 - y^2)^{1/2}}. \tag{7}$$

The integral in Eq. (7) can be evaluated as a gaussian sum, with remainder R_n [1, Sect. 25.4.38].

$$\int_{-1}^{+1} \frac{F(y)}{(1 - y^2)^{1/2}}\, dy = \frac{\pi}{2} \sum_{i=1}^{i=n} F(y_i) + R_n, \tag{8}$$

where

$$F(y_i) = [\cos(j + 1/2)\, u_i]\, L(y_i),$$

$$u_i = \pi(i - 1/2)/n,$$

and

$$y_i = \cos u_i.$$

For $f(x)$ given by Eq. (6), a comparison was made between a_j and A_j, using the explicit forms for the Legendre polynomials for $j \leqslant 12$ given in [1]. For n equal to 10^3 in Eq. (8), the discrepancy between the two values was $< 10^{-6}$. For n equal to 2.10^3, the discrepance was $< 10^{-8}$.

4. Discussion

In Section 3, good agreement was found between the values of A_j calculated numerically from Eq. (7), and a_j, evaluated analytically for values of $j \leqslant 12$. The value of a_j can also be determined numerically from Eq. (1) by a Gauss–Legendre sum [1],

$$a_j = \sum_{i=1}^{n} \omega_i f(x_i) P_j(x_i), \tag{9}$$

where the $P_j(x_i)$ are calculated from the recursion relation,

$$P_j(x_i) = \frac{(2j-1)}{j} x_i P_{j-1}(x_i) - \frac{(j-1)}{j} P_{j-2}(x_i). \tag{10}$$

Alternatively, a_j can be calculated from,

$$a_j = \sum_{i=1}^{n} f(x_i) P_j(x_i) dy_i. \tag{11}$$

For a specified number of points n, the coefficients of $f(x)$ in Eqs. (9) and (11) can be calculated and stored in an array. The computation of a_j then requires n multiplications and n additions. A similar procedure can be used for the evaluation of A_j through Eq. (7) except that, in the latter case, $L(y_i)$ must be obtained instead of $f(x_i)$. However, for a fixed number of points, the accuracy of the coefficients (a_j or A_j) decreases with increasing j value.

If it is desired to calculate all coefficients to the same predetermined accuracy, an adaptive gaussian routine can be used. In such a routine, the subdivision of interval of integration, and the corresponding choices of x_i and y_i, are determined by the specific integral that is being calculated. Consequently, different x_i and y_i values will be selected for different j coefficients. In this situation, it is not feasible to precalculate an array of multiplying factors for $f(x_i)$ or $L(y_i)$ in Eq. (4) or (7).

For the computation of the expansion coefficients to a prescribed confidence level by an adaptive gaussian routine, there are cases where it is advantageous to calculate A_j instead of a_j, particularly for large j values. The calculation of a_j, using Eq. 9, requires the value of $P_j(x_i)$, using Eq. (10), at each gaussian point selected. In turn, this requires all $P_k(x_i)$ values for $k < j$. Each evaluation of Eq. (10) requires 6 multiplications/divisions and 3 additions/subtractions and the number of computations for each $P_j(x_i)$ value can become large if j is large. In contrast, in the calculation of A_j using Eq. (7), the factor multiplying $L(y_i)$ is evaluated only once for each gaussian y_i point. Even though an arc-cosine and cosine are required for this factor, the calculation can be much faster than j evaluations of Eq. (10). Since $L(y_i)$ must also be determined, the greatest advantage ensues when $L(y)$ can be evaluated analytically, as in the example given in Section 3. When $L(y)$ must be evaluated numerically, the advantage is less. The integral in Eq. (4) can be written in a simple gaussian form with the substitution $u^2 = (1 - y^2)$. However, the additional number of computations for each y_i value and the increase in complexity make this procedure generally less attractive.

For the example given in Section 3, an adaptive gaussian routine was used to compute both a_j and A_j to an accuracy of 1%. For $j = 10$, a_j required 207 points and A_j required 215 points, with corresponding times of 24.5 ms and 26.2 ms. However, for $j = 100$, the numbers of points were 699 and 727, while the time for a_j was 647 ms compared with 81 ms for A_j. For $j = 200$, the numbers of points were still approximately equal (~ 1350) but the time for the calculation of a_j was fifteen times longer than for A_j (2.29 s compared with 150 ms).

5. Summary

A new expression has been presented for the Legendre polynomial-expansion coefficients of a function $f(x)$. The polynomials themselves do not appear explicitly in the expression. In addition to its intrinsic interest, the new expression has some interesting features that make it particularly useful for the computation of high-order coefficients for those cases in which $\int f(x)\,dx/(y-x)^{1/2}$ can be evaluated analytically.

Acknowledgment

This work was financially supported, in part, by the Natural Sciences and Engineering Research Council of Canada.

References

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Received: November 25, 1985; revised: June 2, 1986

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